

# COHEN-MACAULAYNESS AND GORENSTEINNESS OF SYMBOLIC BLOWUP ALGEBRAS OF CERTAIN MONOMIAL CURVES

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ABSTRACT. In this paper we answer a question posed by S. Goto in [5]. Let  $d \geq 2, m \geq 1$  and  $\gcd(d, m) = 1$ . In this paper we show that the symbolic blowup algebras  $\mathcal{R}_s(\mathfrak{p})$  and  $G_s(\mathfrak{p})$  corresponding to the monomial curve parameterised by  $x_i \rightarrow t^{n_i}$  where  $1 \leq i \leq d$  and  $n_i = d + (i - 1)m$  are Cohen-Macaulay. We also discuss when these algebras are Gorenstein.

## 1. INTRODUCTION

Let  $(R, \mathfrak{M})$  be a regular local ring and  $\mathfrak{p}$  a prime ideal. Let  $\mathcal{R}_s(\mathfrak{p}) := \bigoplus_{n \geq 0} \mathfrak{p}^{(n)}$  denote the symbolic Rees algebra where  $\mathfrak{p}^{(n)} := \mathfrak{p}^n R_{\mathfrak{p}} \cap R$  and let  $G_s(\mathfrak{p}) := \bigoplus_{n \geq 0} \frac{\mathfrak{p}^{(n)}}{\mathfrak{p}^{(n+1)}}$ . We call these algebras the symbolic blowup algebras. Interest in the properties of the symbolic Rees algebra began with a remarkable result of Cowsik where he proved: Let  $(R, \mathfrak{m})$  be a regular local ring and  $\mathfrak{p}$  a prime ideal such that  $\dim(R/\mathfrak{p}) = 1$ . If  $\mathcal{R}_s(\mathfrak{p})$  is Noetherian, then  $\mathfrak{p}$  is a set-theoretic complete intersection [3]. However, the converse need not be true [3]. Motivated by Cowsik's result, in 1987, Huneke gave necessary and sufficient conditions for  $\mathcal{R}_s(\mathfrak{p})$  to be Noetherian when  $\dim R = 3$  [9]. Huneke's result was generalised in 1991 for  $\dim R \geq 3$  by Morales [11]. All these results paved a new way to study the famous problem of set-theoretic complete intersection.

In 1994 Goto, Nishida and Watanabe used Huneke's result to give a class of monomial curves in  $R = \mathbb{k}[[x, y, z]]$  such that  $\mathcal{R}_s(\mathfrak{p})$  was Noetherian but not Cohen-Macaulay in positive characteristic. They also showed that for these curves  $\mathcal{R}_s(\mathfrak{p})$  was not Noetherian in characteristic zero [7].

Consider the following class of monomial curves: Let  $d \geq 2$ . Let  $R = \mathbb{k}[[x_1, \dots, x_d]]$  and  $S = \mathbb{k}[[t]]$  be formal power series rings over a field  $\mathbb{k}$ . For any positive integer  $m \geq 1$  with  $\gcd(d, m) = 1$ , we put  $n_i := d + (i - 1)m$  for  $i = 1, \dots, d$ . Let  $\mathcal{C}(n_1, \dots, n_d)$  be the affine curve parameterised by  $(t^{n_1}, \dots, t^{n_d})$  and let  $I_{\mathcal{C}(n_1, \dots, n_d)}$  be the ideal defining this monomial curve. In other words, let  $\phi : R \rightarrow \mathbb{k}[[t]]$  denote the homomorphism defined by  $\phi(x_i) = t^{n_i}$  for  $1 \leq i \leq d$  and  $\mathfrak{p} := \ker(\phi) = I_{\mathcal{C}(n_1, \dots, n_d)}$ . It is well known that these curves are a set-theoretic complete intersection (see [2] for  $d = 4$  and [14], [12] for  $d \geq 3$ ). For these curves, Goto found nice elements  $f_i \in \mathfrak{p}^{(i)}$  which satisfy Huneke's criterion for the Noetherianity of  $\mathcal{R}_s(\mathfrak{p})$

2010 *Mathematics Subject Classification*. Primary: 13A30, 13O5, 13H15, 13P10.

*Key words and phrases*. Symbolic Rees algebra, Cohen-Macaulay, Gorenstein.

Both authors were partially funded by a grant from Infosys Foundation.

The second author was supported by INdAM COFOUND Fellowships cofounded by Marie Curie actions, Italy.

[5, Theorem 7.4]. He showed that  $\sqrt{(f_1, \dots, f_{d-1})} = \mathfrak{p}$  [5, Proposition 5.3], thus giving an alternate proof to the fact that  $\mathfrak{p}$  is a set-theoretic complete intersection. He further showed that if  $d = 3, 4$ , then  $\mathcal{R}_s(\mathfrak{p})$  is Cohen-Macaulay [5, Proposition 7.6] and raised the question whether  $\mathcal{R}_s(\mathfrak{p})$  is Cohen-Macaulay for  $d \geq 5$  [5, page 58]. One of the main results in this paper gives a positive answer to Goto's question.

Throughout this paper  $\mathfrak{p} = I_{\mathcal{C}(n_1, \dots, n_d)}$  unless otherwise specified. It is well known that  $\mathfrak{p}$  is generated by the  $2 \times 2$  minors of the matrix described in (2.1). In [5, Proposition 7.6], Goto described  $\mathfrak{p}^{(n)}$  for  $d = 4$  and  $n = 2, 3$ . It is not easy to describe the ideals  $\mathfrak{p}^{(n)}$  in general. To achieve this, we define ideals  $\mathcal{I}_n R \subseteq \mathfrak{p}^{(n)}$  ((2.12)). We exploit the fact that the ideals  $(\mathcal{I}_n, x_1)T$  are homogeneous ideals (Proposition 2.13).

Using the description of  $\mathfrak{p}^{(n)}$  for  $d = 4$  and  $n = 2, 3$ , Goto proved that the rings  $R/(\mathfrak{p}^{(n)} + (f_1, f_2, f_3))$  are Cohen-Macaulay, where the  $f_i \in \mathfrak{p}^{(i)}$  ( $1 \leq i \leq 3$ ) are as described in [5, page 57]. However, from their method it is not easy to prove a similar result for  $d \geq 5$ . The new idea in this paper is to give an ordering on  $T' = T/(x_1)$  which we call the grevlex (Definition 3.1). For any ideal  $I \subseteq T'$ , let  $LI(I)$  denote the leading ideal of  $I$  with respect to the grevlex order. We define monomial ideals  $I_n \subseteq LI(\mathcal{I}_n T')$  ((3.3), Proposition 3.4) and consider the filtration  $\mathcal{F} = \{I_n\}_{n \geq 0}$ . One of the key steps is to prove that the associated graded ring  $G(\mathcal{F}) = \bigoplus_{n \geq 0} (I_n/I_{n+1})$  is Cohen-Macaulay and that  $(x_2^2)^*, \dots, (x_d^d)^*$  is a regular sequence in  $G(\mathcal{F})$  (Theorem 4.6). We next describe the monomials which span the vector space  $I_{n-1}/(I_n : x_d)$  (Proposition 5.3). These computations are very crucial as they enable us to compute  $\ell(T'/I_n)$  and  $\ell(T'/I_n + (x_2^2, \dots, x_{k+1}^{k+1}))$  for  $k = 1, \dots, d-1$  and  $n \geq 1$ . They in turn help us to compute the lengths of the rings  $R/(\mathfrak{p}^{(n)} + (x_1))$  and  $R/(\mathfrak{p}^{(n)} + (x_1, f_1, \dots, f_k))$  where the  $f_i \in \mathfrak{p}^{(i)}$  are as described in [5, page 57]. In particular we prove that for all  $n \geq 1$  and  $k = 1, \dots, d-1$ , the rings  $R/(\mathfrak{p}^{(n)} + (f_1, \dots, f_k))$  are all Cohen-Macaulay. As a byproduct of our computations, we describe the generators of the ideals  $\mathfrak{p}^{(n)}$  and the leading ideals  $LI(\mathfrak{p}^{(n)} T')$  for all  $n \geq 1$  (Proposition 6.8).

If  $d = 3$ , then  $\text{ht}(\mathfrak{p}) = 2$ . Hence, if  $\mathcal{R}_s(\mathfrak{p})$  is Cohen-Macaulay, then it is also Gorenstein ([13, Corollary 3.4]). In [6], Goto et al study the Gorenstein property of the symbolic Rees algebra. In this paper we prove that for  $\mathcal{R}_s(\mathfrak{p})$  is Gorenstein if and only if  $d = 3$  (Theorem 7.2). We also prove that  $G_s(\mathfrak{p})$  is Cohen-Macaulay and Gorenstein (Theorem 7.1).

We now describe the organisation of this paper. In Section 2 we prove some preliminary results which will be needed in the subsequent sections. In Section 3 we describe the ordering we are using and describe the ideals  $I_n T' \subseteq LI(\mathcal{I}_n T')$ . Section 4 is mainly devoted to show that the associated graded ring corresponding to the filtration  $\{I_n\}_{n \geq 0}$  is Cohen-Macaulay. In Section 5 we explicitly describe the monomials which span  $I_{n-1}$  modulo  $(I_n : x_d)$ . Our main results are in Section 6 and Section 7.

#### ACKNOWLEDGEMENTS

The authors would like to thank Prof. J. K. Verma for suggesting the problem and for many useful conversations. The second author would like to thank the Department of Atomic Energy, Government of India, and the Chennai Mathematical Institute (CMI) for providing financial support for her post doctoral

studies at the Institute of Mathematical Sciences (IMSc) and CMI, respectively, during which part of the work is done.

## 2. PRELIMINARIES

**2.1. Computation of multiplicity.** Let  $R = \mathbb{k}[[x_1, \dots, x_d]]$  and  $X = [X_{ij}]$  be the  $d \times d$  matrix given by

$$X_{ij} := \begin{cases} x_{i+j-1} & \text{if } 1 \leq j \leq d-i+1 \\ x_1^m x_{i+j-d-1} & \text{if } d-i+2 \leq j \leq d. \end{cases} \quad (2.1)$$

For each  $1 \leq i, k \leq d-1$ , let  $X(i)$  be the matrix consisting of the first  $i+1$  rows and  $i+1$  columns of  $X$ . We define

$$f_i := \det(X(i)) \quad \text{and} \quad \mathbf{f}_k := f_1, \dots, f_k. \quad (2.2)$$

Goto showed that  $\mathbf{f}_{d-1}$  satisfies Huneke's criterion for the Noetherianness of  $\mathcal{R}_s(\mathbf{p})$  ([5, Theorem 7.4]).

In this section we give a lower bound for the length of the modules  $R/(\mathbf{p}^{(n)} + (x_1, \mathbf{f}_k))$  where  $\mathbf{p} = I_{\mathcal{C}(n_1, \dots, n_d)}$ ,  $1 \leq k \leq d-1$  and  $n \geq 1$ . We need a few preliminary results.

Let  $(A, \mathbf{n})$  be a Noetherian local ring of positive dimension  $d$  and  $\mathfrak{a}$  an  $\mathbf{n}$ -primary ideal. Let  $\mathcal{F} = \{\mathcal{F}(n)\}_{n \in \mathbb{Z}}$  be a Noetherian filtration of ideals, i.e.,  $\mathcal{F}(0) = A$ ,  $\mathcal{F}(1) \neq A$ ,  $\mathcal{F}(n+1) \subseteq \mathcal{F}(n)$ ,  $\mathcal{F}(n) \cdot \mathcal{F}(m) \subseteq \mathcal{F}(n+m)$  for all  $n, m \in \mathbb{Z}$  and the Rees ring  $\mathcal{R}(\mathcal{F}) := \bigoplus_{n \geq 0} \mathcal{F}(n)t^n$  is Noetherian. Let  $1 \leq k \leq d$  and  $z_i \in \mathcal{F}(a_i) \setminus \mathcal{F}(a_i+1)$  for all  $i = 1, \dots, k$ . Put  $\mathbf{z}_k = z_1, \dots, z_k$ . For all  $n \in \mathbb{Z}$ , using the mapping cone construction, similar to that in [8], we construct the complex  $C_\bullet(\mathbf{z}_k; n)$  which has the form:

$$0 \longrightarrow \frac{A}{\mathcal{F}(n - (a_1 + \dots + a_k))} \longrightarrow \dots \longrightarrow \bigoplus_{1 \leq i < j \leq k} \frac{A}{\mathcal{F}(n - a_i - a_j)} \longrightarrow \bigoplus_{i=1}^k \frac{A}{\mathcal{F}(n - a_i)} \longrightarrow \frac{A}{\mathcal{F}(n)} \longrightarrow 0. \quad (2.3)$$

The maps are from the Koszul complex  $K_\bullet(\mathbf{z}_k, A)$ . Let  $H_i(C_\bullet(\mathbf{z}_k, n))$  denote the  $i$ -th homology of the complex  $C_\bullet(\mathbf{z}_k; n)$ .

For any element  $z \in \mathcal{F}(n) \setminus \mathcal{F}(n+1)$ , let  $z^*$  denote the image of  $z$  in  $G(\mathcal{F}) := \bigoplus_{n \in \mathbb{N}} \mathcal{F}(n)/\mathcal{F}(n+1)$ . Let  $\mathbf{z}_k^* := z_1^*, \dots, z_k^*$ .

**Proposition 2.4.** *For  $1 \leq i \leq k$ , let  $z_i \in \mathcal{F}(a_i) \setminus \mathcal{F}(a_i+1)$ . Suppose  $\mathbf{z}_k^*$  is a regular sequence in  $G(\mathcal{F})$ . Then*

- (1)  $H_i(C_\bullet(\mathbf{z}_k, n)) = 0$  for all  $i \geq 1$  and all  $n \in \mathbb{Z}$ .
- (2)  $\ell\left(\frac{A}{\mathcal{F}(n) + (\mathbf{z}_k)}\right) = \sum_{i=0}^k (-1)^i \left[ \sum_{1 \leq j_1 < \dots < j_i \leq k} \ell\left(\frac{A}{\mathcal{F}(n - [a_{j_1} + \dots + a_{j_i}])}\right) \right].$

*Proof.* (1) Let  $K_\bullet(\mathbf{z}_k^*, G(\mathcal{F}))$  denote the Koszul complex of  $G(\mathcal{F})$  with respect to  $\mathbf{z}_k^*$ . Then we have the short exact sequence of complexes:

$$0 \longrightarrow K_\bullet(\mathbf{z}_k^*, G(\mathcal{F}))_{n-1} \longrightarrow C_\bullet(\mathbf{z}_k, n) \longrightarrow C_\bullet(\mathbf{z}_k, n-1) \longrightarrow 0. \quad (2.5)$$

Since  $\mathbf{z}_k^*$  is a regular sequence in  $G(\mathcal{F})$ ,  $H_i(K_\bullet(\mathbf{z}_k^*, G(\mathcal{F}))) = 0$  for all  $i \geq 1$  [10, Theorem 16.5]. Hence from (2.5) for all  $n \in \mathbb{Z}$  we have:

$$H_i(C_\bullet(\mathbf{z}_k, n)) \cong H_i(C_\bullet(\mathbf{z}_k, n-1)) \quad \text{for all } i \geq 2$$

and the short exact sequence

$$0 \longrightarrow H_1(C_\bullet(\mathbf{z}_k, n)) \longrightarrow H_1(C_\bullet(\mathbf{z}_k, n-1)).$$

As  $H_i(C_\bullet(\mathbf{z}_k, n)) = 0$  for all  $n \leq 0$ , we conclude that  $H_i(C_\bullet(\mathbf{z}_k, n)) = 0$  for all  $n$  and for all  $i \geq 1$ . This proves (1).

(2) As  $H_0(C_\bullet(\mathbf{z}_k, n)) = A/(\mathcal{F}(n) + (\mathbf{z}_k))$ , from the complex (2.3) we get

$$\ell\left(\frac{A}{\mathcal{F}(n) + (\mathbf{z}_k)}\right) + \sum_{i \geq 1} (-1)^i \ell(H_i(C_\bullet(\mathbf{z}_k, n))) = \sum_{i=0}^k (-1)^i \left[ \sum_{1 \leq j_1 < \dots < j_i \leq k} \ell\left(\frac{A}{\mathcal{F}(n - [a_{j_1} + \dots + a_{j_i}])}\right) \right].$$

Applying (1) we get the result.  $\square$

**Corollary 2.6.** *Let  $(A, \mathfrak{n})$  be a Cohen-Macaulay local ring of dimension  $d$ . Let  $\mathfrak{p}$  be a prime ideal of height  $d-1$  and  $x \notin \mathfrak{p}$ . Let  $1 \leq k \leq d-1$  and  $z_i \in \mathfrak{p}^{(a_i)} \setminus \mathfrak{p}^{(a_i+1)}$ . Suppose  $\mathbf{z}_k^*$  is a regular sequence in  $G(\mathfrak{p}A_{\mathfrak{p}})$ . Then*

$$\begin{aligned} (1) \quad e\left(x; \frac{A}{\mathfrak{p}^{(n)} + (\mathbf{z}_k)}\right) &= \ell\left(\frac{A}{(\mathfrak{p}, x)}\right) \sum_{i=0}^k (-1)^i \left[ \sum_{1 \leq j_1 < \dots < j_i \leq k} \ell\left(\frac{A_{\mathfrak{p}}}{\mathfrak{p}^{n-[a_{j_1} + \dots + a_{j_i}]} A_{\mathfrak{p}}}\right) \right]. \\ (2) \quad \ell\left(\frac{A}{\mathfrak{p}^{(n)} + (\mathbf{z}_k) + (x)}\right) &\geq \ell\left(\frac{A}{(\mathfrak{p}, x)}\right) \sum_{i=0}^k (-1)^i \left[ \sum_{1 \leq j_1 < \dots < j_i \leq k} \ell\left(\frac{A_{\mathfrak{p}}}{\mathfrak{p}^{n-[a_{j_1} + \dots + a_{j_i}]} A_{\mathfrak{p}}}\right) \right]. \end{aligned}$$

*Proof.* (1) As  $\mathfrak{p}^{(n)} \subseteq \mathfrak{p}^{(n)} + (\mathbf{z}_k) \subseteq \mathfrak{p}$ , taking radicals we get  $\sqrt{\mathfrak{p}^{(n)} + (\mathbf{z}_k)} = \mathfrak{p}$ . Hence  $\mathfrak{p}$  is the only minimal prime of  $\mathfrak{p}^{(n)} + (\mathbf{z}_k)$ . From the associativity formula for multiplicities [10, Theorem 14.7] we get

$$e\left(x; \frac{A}{\mathfrak{p}^{(n)} + (\mathbf{z}_k)}\right) = e\left(x; \frac{A}{\mathfrak{p}}\right) \ell\left(\frac{A_{\mathfrak{p}}}{(\mathfrak{p}^{(n)} + (\mathbf{z}_k))A_{\mathfrak{p}}}\right).$$

As  $x$  is a nonzero divisor on  $A/\mathfrak{p}$ ,  $e(x; A/\mathfrak{p}) = \ell(A/(\mathfrak{p}, x))$ . Replacing  $A$  by  $A_{\mathfrak{p}}$  and  $G(\mathcal{F})$  by  $G(\mathfrak{p}A_{\mathfrak{p}})$  in Proposition 2.4(2) we get the result.

(2) From [10, Theorem 14.10], we get  $\ell\left(\frac{A}{\mathfrak{p}^{(n)} + (\mathbf{z}_k) + (x)}\right) \geq e\left(x; \frac{A}{\mathfrak{p}^{(n)} + (\mathbf{z}_k)}\right)$ . Now apply (1).  $\square$

**Theorem 2.7.** *Let  $R = \mathbb{k}[[x_1, \dots, x_d]]$  and  $\mathfrak{p} = I_{C(n_1, \dots, n_d)}$ . For  $1 \leq i, k \leq d-1$ , let  $f_i$  and  $\mathbf{f}_k$  be as in (2.2). Then*

$$\ell\left(\frac{R}{\mathfrak{p}^{(n)} + (\mathbf{f}_k) + (x_1)}\right) \geq \ell\left(\frac{R}{(\mathfrak{p}, x)}\right) \sum_{i=0}^k (-1)^i \left[ \sum_{1 \leq j_1 < \dots < j_i \leq k} \ell\left(\frac{R_{\mathfrak{p}}}{\mathfrak{p}^{n-[j_1 + \dots + j_i]} R_{\mathfrak{p}}}\right) \right].$$

*Proof.* By [5, Lemma 7.5],  $f_i \in \mathfrak{p}^{(i)}$ . As  $G(\mathfrak{p}R_{\mathfrak{p}})$  is a regular ring and  $\mathbf{f}_{d-1}^*$  is a regular sequence [5, Proposition 5.3(3)], from Corollary 2.6(2), we get the result.  $\square$

**2.2. The power series ring and the polynomial ring.** From now on  $R = \mathbb{k}[[x_1, \dots, x_d]]$  and  $T = \mathbb{k}[x_1, \dots, x_d]$ . The following lemma gives us a way to compute the length of an  $R$ -module in terms of the length of the corresponding  $T$ -module.

**Lemma 2.8.** *Let  $\mathfrak{m} = (x_1, \dots, x_d)T$  and  $M$  a finitely generated  $T$ -module such that  $\text{Supp}(M) = \{\mathfrak{m}\}$ . Then*

$$\ell_R(M \otimes_T R) = \ell_T(M).$$

*Proof.* We prove by induction on  $\ell_T(M)$ . If  $\ell_T(M) = 1$ , then  $M \cong T/\mathfrak{m}$ . Therefore,

$$\ell_R(M \otimes_T R) = \ell_R\left(\frac{R}{\mathfrak{m}R}\right) = 1 \quad (\text{as } \mathfrak{m}R \text{ is the maximal ideal of } R).$$

If  $\ell_T(M) > 1$ , then as the minimal primes of  $\text{Supp}(M)$  and  $\text{Ass}(M)$  are the same,  $\mathfrak{m} \in \text{Ass}(M)$ . This gives the exact sequence

$$0 \longrightarrow \frac{T}{\mathfrak{m}} \longrightarrow M \longrightarrow C \longrightarrow 0, \quad (2.9)$$

where  $C \cong M/(T/\mathfrak{m})$ . As  $R$  is  $T$ -flat, tensoring (2.9) with  $R$  we get:

$$0 \longrightarrow \frac{T}{\mathfrak{m}} \otimes_T R \cong \frac{R}{\mathfrak{m}R} \longrightarrow M \otimes_T R \longrightarrow C \otimes_T R \longrightarrow 0.$$

From the exact sequence (2.9), we get  $\text{Supp}(C) = \{\mathfrak{m}\}$  and  $\ell_T(C) < \ell_T(M)$ . Therefore by induction hypothesis  $\ell_R(C \otimes_T R) = \ell_T(C)$ . Hence

$$\ell_R(M \otimes_T R) = \ell_R(C \otimes_T R) + \ell_R\left(\frac{R}{\mathfrak{m}R}\right) = \ell_T(C) + \ell_T\left(\frac{T}{\mathfrak{m}}\right) = \ell_T(M).$$

□

Let  $X_{i+1, (j_1, \dots, j_{i+1})}$  be the matrix obtained by choosing the first  $i+1$  rows and  $j_1, \dots, j_{i+1}$  columns of  $X$  and let

$$\mathcal{J}_i = \{\det(X_{i+1, (j_1, \dots, j_{i+1})}) \mid 1 \leq j_1 < \dots < j_{i+1} \leq d\}. \quad (2.10)$$

**Notation 2.11.** If  $A_1, \dots, A_n$  are  $n$  sets of monomials we define the set  $A_1 \cdots A_n$  by  $A_1 \cdots A_n := \{a_1 \cdots a_n : a_i \in A_i\}$ .

Let  $\mathcal{I}_n$  denote the set

$$\mathcal{I}_n := \sum_{a_1 + 2a_2 + \dots + (d-1)a_{d-1} = n} \mathcal{J}_1^{a_1} \cdots \mathcal{J}_{d-1}^{a_{d-1}}. \quad (2.12)$$

As  $R$  is a flat  $T$ -module,  $\mathcal{I}_n R = \mathcal{I}_n T \otimes_T R$ .

**Proposition 2.13.** *Let  $n \geq 1$ . Then*

- (1)  $\mathcal{I}_n R \subseteq \mathfrak{p}^{(n)}$ .
- (2)  $(\mathcal{I}_n + (x_1))T$  is an homogeneous ideal.

(3)  $(\mathcal{I}_n + (x_1))T$  is an  $\mathfrak{m}$ -primary ideal.

*Proof.* (1) By [5, Lemma 7.5],  $\mathcal{J}_i \subseteq \mathfrak{p}^{(i)}$  for all  $i = 1, \dots, d-1$ . Hence for all  $a_1, \dots, a_{d-1} \in \mathbb{Z}_{\geq 0}$ ,

$$\mathcal{J}_1^{a_1} \dots \mathcal{J}_{d-1}^{a_{d-1}} \subseteq \mathfrak{p}^{a_1} (\mathfrak{p}^{(2)})^{a_2} \dots (\mathfrak{p}^{(d-1)})^{a_{d-1}} \subseteq \mathfrak{p}^{(a_1 + 2a_2 + \dots + (d-1)a_{d-1})}. \quad (2.14)$$

Summing over all  $a_1 + 2a_2 + \dots + (d-1)a_{d-1} = n$  and applying (2.14) to (2.12) we get (1).

(2) Fix  $1 \leq j_1 < j_2 < \dots < j_{i+1} \leq d$ . Then  $\det(X_{i+1, (j_1, \dots, j_{i+1})})$  is a sum of distinct monomials and the monomials which do not contain  $x_1$  are homogeneous of degree  $i+1$ . Hence  $(\mathcal{J}_i + (x_1))T$  is an homogeneous ideal. From (2.12) we get (2).

(3) By (2.12),  $\mathcal{J}_1^n \subseteq \mathcal{I}_n$  and  $\mathcal{J}_1^n + (x_1) = (x_2, \dots, x_d)^{2n} + (x_1)$  which implies that  $\mathfrak{m} = \sqrt{\mathcal{J}_1^n + (x_1)} \subseteq \sqrt{\mathcal{I}_n + (x_1)} \subseteq \mathfrak{m}$ .  $\square$

### 3. MONOMIAL ORDERING AND INITIAL IDEALS

Let  $T' = \mathbb{k}[x_2, \dots, x_d] \cong T/x_1T$ . We put the grevlex monomial order on monomials in  $T'$  which is defined as follows:

**Definition 3.1.** Let  $\mathbf{a} = (a_2, \dots, a_d)$  and  $\mathbf{x}^{\mathbf{a}} := \prod_{i=2}^d x_i^{a_i}$ . We say that  $\mathbf{x}^{\mathbf{a}} > \mathbf{x}^{\mathbf{b}}$  if  $\deg(\mathbf{x}^{\mathbf{a}}) > \deg(\mathbf{x}^{\mathbf{b}})$  or  $\deg(\mathbf{x}^{\mathbf{a}}) = \deg(\mathbf{x}^{\mathbf{b}})$  and in the ordered tuple  $(a_2 - b_2, \dots, a_d - b_d)$  the left-most nonzero entry is negative.

Note that with respect to this order we have  $x_2 < x_3 < \dots < x_d$ . For any polynomial  $f \in T'$ , let  $LM(f)$  denote the initial term of  $f$  and for any ideal  $I \subset T'$ , let  $LI(I)$  be the initial ideal of the ideal  $I$  with respect to the grevlex order. Our aim in this section is to describe some of the terms of the leading ideal of  $\mathcal{I}_n T'$ .

For  $2 \leq r < s \leq d$  and  $l \geq 1$ , let  $M_{r,s}^l$  denote the set of monomials of degree  $l$  in the variables  $x_r, \dots, x_s$ . We set  $M_{r,s} := M_{r,s}^1$ .

Let  $1 \leq i \leq d-1$  and  $n \geq 1$ . We define the ideals  $J_i$  and  $I_n$  in  $T'$  as follows:

$$J_i := (M_{i+1,d})^{i+1}, \quad (3.2)$$

$$I_n := \sum_{a_1 + 2a_2 + \dots + (d-1)a_{d-1} = n} J_1^{a_1} \dots J_{d-1}^{a_{d-1}}. \quad (3.3)$$

**Proposition 3.4.** For all  $n \geq 1$ ,  $I_n \subseteq LI(\mathcal{I}_n T')$ .

To prove Proposition 3.4, we first need to consider  $LI(\det(X_{i+1, (j_1, \dots, j_{i+1})}))$  for all  $1 \leq j_1 < \dots < j_{i+1} \leq d$ . This is done in Proposition 3.6.

**Notation 3.5.** For any  $n \times n$  matrix  $M = (m_{ij})$ , let  $p(M) := \prod_{i+j=n+1} m_{ij}$  denote the product of anti-diagonal elements of the matrix  $M$ .

**Proposition 3.6.** For  $1 \leq i \leq d$ ,

- (1)  $p(X_{i+1,(j_1,\dots,j_{i+1})}) = LM(\det(X_{i+1,(j_1,\dots,j_{i+1})})T') = \prod_{k=1}^{i+1} x_{j_k+(i-k+1)}.$   
 (2)  $J_i \subseteq LI(\mathcal{J}_i T').$

*Proof.* (1) By definition,  $p(X_{i+1,(j_1,\dots,j_{i+1})}) = \prod_{k=1}^{i+1} X_{i-k+2,j_k}.$

We claim that  $X_{i-k+2,j_k} = x_{j_k+(i-k+1)}$  for all  $k = 1, \dots, i+1$ . Since  $j_k \leq j_{i+1} - (i-k+1)$  for all  $1 \leq k \leq i+1$ , it follows that  $j_k + (i-k+2) \leq j_{i+1} + 1 \leq d+1$ . Hence the matrix  $X_{i+1,(j_1,\dots,j_{i+1})}$  is

$$\begin{pmatrix} x_{j_1} & \cdots & x_{j_k} & \cdots & X_{1,j_{i+1}} = x_{j_{i+1}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ x_{j_1+(i-k+1)} & \ddots & X_{i-k+2,j_k} = x_{j_k+(i-k+1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ X_{i+1,j_1} = x_{j_1+i} & \ddots & \ddots & \ddots & \star \end{pmatrix}. \quad (3.7)$$

This proves the claim. Hence  $p(X_{i+1,(j_1,\dots,j_{i+1})}) = \prod_{k=1}^{i+1} x_{j_k+(i-k+1)}.$

To complete the proof of (1) we need to show that:

$$LM(\det(X_{i+1,(j_1,\dots,j_{i+1})})T') = \prod_{k=1}^{i+1} x_{j_k+(i-k+1)}. \quad (3.8)$$

We prove (3.8) by induction on  $i$ . Note that in the matrix  $X$  defined in (2.1),  $x_1$  divides  $X_{11}$  and  $X_{ij}$  for  $i+j \geq d+2$ . Let  $i = 1$ . Then

$$\det(X_{2,(j_1,j_2)})T' = \begin{cases} x_{j_1+1}x_{j_2} & \text{if } j_1 = 1 \text{ or } j_2 = d \\ x_{j_1}x_{j_2+1} - x_{j_1+1}x_{j_2} & \text{if } 1 < j_1 < j_2 < d. \end{cases}$$

Hence  $LM(\det(X_{2,(j_1,j_2)})T') = x_{j_1+1}x_{j_2}$  if  $j_1 = 1$  or  $j_2 = d$ . If  $1 < j_1 < j_2 < d$ , then  $j_1 < j_1+1 \leq j_2 < j_2+1$  and hence  $x_{j_1+1}x_{j_2} > x_{j_1}x_{j_2+1}$  which implies that  $LM(\det(X_{2,(j_1,j_2)})T') = x_{j_1+1}x_{j_2}$ . Hence (3.8) is true for  $i = 1$ .

Now let  $i > 1$ . Expanding the matrix in (3.7) along the last row we get:

$$\det(X_{i+1,(j_1,\dots,j_{i+1})})T' = \left( \sum_{k=1}^t (-1)^{k+i+1} x_{j_k+i} \det(X_{i,(j_1,\dots,\widehat{j_k},\dots,j_{i+1})}) \right) T' \quad (3.9)$$

where  $t = \max\{k | j_k + i \leq d\}$ . As  $X_{i,(j_1,\dots,\widehat{j_k},\dots,j_{i+1})}$  has the form as the matrix described in (3.7), by induction hypothesis,

$$\begin{aligned} LM(\det(X_{i,(j_1,\dots,\widehat{j_k},\dots,j_{i+1})})T') &= \prod_{\alpha=1}^{k-1} x_{j_\alpha+i-\alpha} \prod_{\alpha=k+1}^{i+1} x_{j_\alpha+(i-\alpha+1)} \\ &= \begin{cases} \prod_{\alpha=2}^{i+1} x_{j_\alpha+(i-\alpha+1)} & \text{if } k = 1 \\ x_{j_1+(i-1)} \prod_{\alpha=2}^{k-1} x_{j_\alpha+i-\alpha} \prod_{\alpha=k+1}^{i+1} x_{j_\alpha+(i-\alpha+1)} & \text{if } k = 2, \dots, t. \end{cases} \end{aligned} \quad (3.10)$$

Hence for all  $k = 2, \dots, t$

$$\begin{aligned} &x_{j_k+i} LM(\det(X_{i,(j_1,\dots,\widehat{j_k},\dots,j_{i+1})})T') \\ &= x_{j_1+(i-1)} \left[ \prod_{\alpha=2}^{k-1} x_{j_\alpha+i-\alpha} \right] x_{j_k+i} \left[ \prod_{\alpha=k+1}^{i+1} x_{j_\alpha+(i-\alpha+1)} \right] \quad [\text{by (3.10)}] \\ &< x_{j_1+i} x_{j_2+(i-1)} \cdots x_{j_i+1} x_{j_{i+1}} \\ &= x_{j_1+i} LM(\det(X_{i,(\widehat{j_1},j_2,\dots,j_{i+1})})T') \quad [\text{by (3.10)}]. \end{aligned} \quad (3.11)$$

Therefore

$$\begin{aligned} LM(\det(X_{i+1,(j_1,\dots,j_{i+1})})T') &= LM\left(\sum_{k=1}^t (-1)^{k+i+1} x_{j_k+i} LM(\det(X_{i,(j_1,\dots,\widehat{j_k},\dots,j_{i+1})})T')\right) \quad [\text{by (3.9)}] \\ &= x_{j_1+i} LM(\det(X_{i,(\widehat{j_1},j_2,\dots,j_{i+1})})T') \quad [\text{by (3.11)}] \\ &= x_{j_1+i} x_{j_2+(i-1)} \cdots x_{j_i+1} x_{j_{i+1}} \quad [\text{by (3.10)}]. \end{aligned}$$

This proves (1).

(2) Let  $x_{i+k_1}^{\alpha_{i+k_1}} \cdots x_{i+k_s}^{\alpha_{i+k_s}} \in M_{i+1,d}^{i+1}$  where  $1 \leq k_1 < k_2 < \cdots < k_s \leq d-i$  and  $\alpha_{i+k_1}, \dots, \alpha_{i+k_s} \neq 0$  such that  $\alpha_{i+k_1} + \cdots + \alpha_{i+k_s} = i+1$ . Set  $\beta_0 = 0$  and  $\beta_r = \alpha_{i+k_1} + \cdots + \alpha_{i+k_r}$  for  $1 \leq r \leq s$ . Define

$$S_r = \{\beta_{r-1} + 1, \beta_{r-1} + 2, \dots, \beta_r\} \text{ for } 1 \leq r \leq s.$$

Then  $\bigsqcup_{r=1}^s S_r = \{1, \dots, i+1\}$ . Let  $1 \leq t \leq i+1$ . If  $t \in S_r$  define

$$j_t = k_r + (t - 1).$$

With this choice of  $j_1, \dots, j_{i+1}$ ,  $p(X_{i+1,(j_1,\dots,j_{i+1})}) = x_{i+k_1}^{\alpha_{i+k_1}} \cdots x_{i+k_s}^{\alpha_{i+k_s}}$ . By (1)  $x_{i+k_1}^{\alpha_{i+k_1}} \cdots x_{i+k_s}^{\alpha_{i+k_s}} \in LI(\mathcal{J}_i T')$ . Hence  $J_i \subseteq LI(\mathcal{J}_i T')$ .  $\square$

**Proof of Proposition 3.4:** The proof follows from (2.12), Proposition 3.6(2) and (3.3).



#### 4. THE ASSOCIATED GRADED RING CORRESPONDING TO THE FILTRATION $\mathcal{F} := \{I_n\}_{n \geq 0}$

Let  $G(\mathcal{F}) := \oplus_{n \geq 0} I_n / I_{n+1}$  be the associated graded ring corresponding to the filtration  $\mathcal{F} = \{I_n\}_{n \geq 0}$ , where  $I_n$  are ideals defined in (3.3). Then, by definition of  $I_n$ ,  $G(\mathcal{F})$  is Noetherian.

In this section we show that  $G(\mathcal{F})$  is Cohen-Macaulay (Theorem 4.6). As an immediate consequence, we give a formula for  $\ell\left(\frac{T'}{(I_n + (x_2^2, \dots, x_k^k))T'}\right)$  (Proposition 4.7) which is useful in the subsequent sections. The following proposition is crucial to prove Theorem 4.6.

**Proposition 4.1.** *For all  $n \geq 1$  and  $i = 2, \dots, d$ ,*

$$(I_n : (x_i^i)) = \begin{cases} T' & \text{if } n < i \\ I_{n-i+1} & \text{if } n \geq i. \end{cases}$$

*Proof.* If  $n < i$ , then  $x_i^i \in J_{i-1} \subseteq I_n$  which implies that  $(I_n : (x_i^i)) = T'$ . Therefore, for the rest of the proof we will assume that  $n \geq i$ .

As  $I_n = \sum_{a_1+2a_2+\dots+(d-1)a_{d-1}=n} J_1^{a_1} \cdots J_{d-1}^{a_{d-1}}$ , by [4, Proposition 1.14], we only need to consider  $M_j \in J_j^{a_j}$

with  $\deg(M_j) = (j+1)a_j$  and show that  $((\prod_{j=1}^{d-1} M_j) : (x_i^i)) \subseteq I_{n-i+1}$ . Note that

$$\left( \left( \prod_{j=1}^{d-1} M_j \right) : (x_i^i) \right) = \left( \frac{(\prod_{j=1}^{d-1} M_j)}{\gcd(\prod_{j=1}^{d-1} M_j, x_i^i)} \right) = \left( \frac{(\prod_{j=1}^{i-1} M_j)}{x_i^g} \left[ \prod_{j=i}^{d-1} M_j \right] \right)$$

where  $g = \min\{i, \sum_{j=1}^{i-1} b_j\}$  and  $b_j := \max\{t | x_i^t \text{ divides } M_j\}$ .

If  $b_j = 0$  for all  $j = 1, \dots, i-1$ , then  $g = 0$  and  $\left( \prod_{j=1}^{d-1} M_j \right) : (x_i^i) = \left( \prod_{j=1}^{d-1} M_j \right) \subseteq I_n \subseteq I_{n-i+1}$ . Hence, for the rest of the proof we will assume that  $b_j \neq 0$  for some  $j = 1, \dots, i-1$ .

**Claim:** For  $j = 1, \dots, i-1$ , there exist integers  $a'_j$  and monomials  $M'_j$  such that:

- (1)  $M'_j \in J_j^{a'_j}$  for all  $j = 1, \dots, i-1$ .
- (2)  $\frac{\left( \prod_{j=1}^{i-1} M_j \right)}{x_i^g} = \left( \prod_{j=1}^{i-1} M'_j \right) N$ , for some monomial  $N$  in  $T'$ .
- (3)  $\sum_{j=1}^{i-1} j a'_j + \sum_{j=i}^{d-1} j a_j \geq n - i + 1$ .

Put  $k := \min \left\{ l \left| \sum_{j=l}^{i-1} b_j \leq i-1 \right. \right\}$ . For  $k \leq j \leq i-1$  we define  $q_j$  and  $r_j$  using the following algorithm:

---

**Algorithm**


---

**Output:** Defines  $q_j, r_j$  for  $k \leq j \leq i - 1$ .

**Initialize:**  $r_i = 0$  and  $j = i - 1$

```

1: while  $j \geq k$  do
2:   if  $b_j = 0$  then
3:     define  $q_j = 0$  and  $r_j = r_{j+1}$ 
4:   else
5:     find integers  $q_j$  and  $0 \leq r_j \leq j$  such that

```

$$b_j - r_{j+1} = (j + 1)q_j - r_j. \quad (4.2)$$

```

6:   end if
7:   return  $q_j, r_j$ 
8:    $j \leftarrow j - 1$ 
9: end while

```

---

Define non-negative integers  $q_{k-1}$  and  $r_{k-1}$  as follows: Put

$$c := g - \sum_{j=k}^{i-1} b_j. \quad (4.3)$$

If  $c = 0$ , then put  $q_{k-1} := 0$  and  $r_{k-1} := r_k$ . If  $c > 0$ , then choose  $q_{k-1} \geq 0$  and  $0 \leq r_{k-1} \leq k - 1$  such that

$$c - r_k = kq_{k-1} - r_{k-1}. \quad (4.4)$$

For  $j = 1, \dots, i - 1$  we define  $a'_j$  as follows:

$$a'_j := \begin{cases} a_j - q_j & \text{if } j \in \{k - 1, \dots, i - 1\} \setminus \{r_{k-1} - 1\} \\ a_j & \text{if } j \in \{1, \dots, k - 2\} \setminus \{r_{k-1} - 1\} \\ a_{r_{k-1}-1} + 1 & \text{if } j = r_{k-1} - 1 \text{ and } r_{k-1} \geq 2. \end{cases} \quad (4.5)$$

Set  $M_0 = 1$  and define  $N_j$  for  $j = k - 1, \dots, i$  as follows:

$$N_j = \begin{cases} 1 & \text{if } j = i \\ \text{a monomial of degree } r_j \text{ that divides } \frac{M_j N_{j+1}}{x_i^{b_j}} & \text{if } b_j \neq 0 \text{ and } k \leq j < i \\ N_{j+1} & \text{if } b_j = 0 \text{ and } k \leq j < i \\ \text{a monomial of degree } r_{k-1} \text{ that divides } \frac{M_{k-1} N_k}{x_i^c} & \text{if } j = k - 1. \end{cases}$$

For  $j = 1, \dots, i-1$  we define  $M'_j$  as follows:

$$M'_j := \begin{cases} \frac{M_j N_{j+1}}{x_i^{b_j} N_j} & \text{if } j \in \{k, \dots, i-1\} \setminus \{r_{k-1}-1\} \\ \frac{M_{k-1} N_k}{x_i^c N_{k-1}} & \text{if } j = k-1 \\ M_j & \text{if } j \in \{1, \dots, k-2\} \setminus \{r_{k-1}-1\} \\ M_{r_{k-1}-1} N_{k-1} & \text{if } j = r_{k-1}-1 \text{ and } 2 \leq r_{k-1}. \end{cases}$$

By our definition of  $M'_j$ ,  $\deg(M'_j) = (j+1)a'_j$  for all  $j = 1, \dots, i-1$ . Hence  $M'_j \in J_j^{a'_j}$ . This proves (1) of the Claim.

Let

$$N = \begin{cases} N_{k-1} & \text{if } r_{k-1} \leq 1 \\ 1 & \text{if } r_{k-1} > 1. \end{cases}$$

Then we can express  $\prod_{j=1}^{i-1} M_j$  as in (2) of the Claim.

We now prove (3) of the Claim. To complete the proof it suffices to show that  $\sum_{j=1}^{i-1} ja'_j + \sum_{j=i}^{d-1} ja_j \geq n - i + 1$ .

Put

$$\alpha(r_{k-1}) := \begin{cases} 0 & \text{if } r_{k-1} = 0 \\ r_{k-1} - 1 & \text{if } r_{k-1} \neq 0. \end{cases}$$

Then

$$\begin{aligned} & \sum_{j=1}^{i-1} ja'_j + \sum_{j=i}^{d-1} ja_j \\ &= n - \sum_{j=k-1}^{i-1} [(j+1)q_j] + \sum_{j=k-1}^{i-1} q_j + \alpha(r_{k-1}) && [\text{by (4.5)}] \\ &= n - [c - r_k + r_{k-1}] - \sum_{j=k}^{i-1} [b_j - r_{j+1} + r_j] + \sum_{j=k-1}^{i-1} q_j + \alpha(r_{k-1}) && [\text{by (4.4) and (4.2)}] \\ &= n - g + [\alpha(r_{k-1}) - r_{k-1}] + \sum_{j=k-1}^{i-1} q_j && [\text{by (4.3)}]. \end{aligned}$$

We claim that:

(a)  $\sum_{j=k-1}^{i-1} q_j \geq 1$ .

(b) If  $g = i$  and  $r_{k-1} > 0$ , then  $\sum_{j=k-1}^{i-1} q_j \geq 2$ .

Suppose  $q_j = 0$  for all  $j = k - 1, \dots, i - 1$ . Then

$$0 \leq g = \sum_{j=k}^{i-1} b_j + c = \sum_{j=k}^{i-1} [r_{j+1} - r_j] + r_k - r_{k-1} = -r_{k-1} \leq 0,$$

which implies that  $g = 0$ . Hence  $\sum_{j=k}^{i-1} b_j = 0$  which leads to a contradiction on our assumption of  $b_j$ 's. This proves (a) of the claim.

Now suppose  $g = i$  and  $r_{k-1} > 0$ . By (a),  $\sum_{j=k-1}^{i-1} q_j \geq 1$ . If  $\sum_{j=k-1}^{i-1} q_j = 1$ , then  $q_l = 1$  for some  $k - 1 \leq l \leq i - 1$  and  $q_j = 0$  for  $j \neq l$ . Hence

$$i = g = \sum_{j=k}^{i-1} b_j + c = (l + 1) - r_{k-1} \leq i - r_{k-1} \leq i - 1$$

which leads to a contradiction.

If  $g \leq i - 1$  or  $g = i$  and  $r_{k-1} = 0$ , then by Claim (a),  $-g + [\alpha(r_{k-1}) - r_{k-1}] + \sum_{j=k-1}^{i-1} q_j \geq -i + 1$ . If  $g = i$  and  $r_{k-1} \neq 0$ , then by Claim (b)  $-g + [\alpha(r_{k-1}) - r_{k-1}] + \sum_{j=k-1}^{i-1} q_j \geq -i + 1$ . This completes the proof of (3) of the Claim.  $\square$

**Theorem 4.6.** *The associated graded ring  $G(\mathcal{F})$  is Cohen-Macaulay.*

*Proof.* Let  $a^*$  denote the image of  $a$  in  $G(\mathcal{F})$ . Since  $x_i^i \in I_{i-1} \setminus I_i$  it follows that  $(x_i^i)^* \in [G(\mathcal{F})]_{i-1}$ . To prove the theorem it is enough to show that  $(x_2^2)^*, \dots, (x_i^i)^*$  is a regular sequence in  $G(\mathcal{F})$  for all  $2 \leq i \leq d$ . We prove by induction on  $i$ . If  $i = 2$ , then by Proposition 4.1,  $(x_2^2)^*$  is a regular element in  $G(\mathcal{F})$ . Now let  $i > 2$  and assume that  $(x_2^2)^*, \dots, (x_{i-1}^{i-1})^*$  is a regular sequence in  $G(\mathcal{F})$ . Then

$$\frac{G(\mathcal{F})}{((x_2^2)^*, \dots, (x_{i-1}^{i-1})^*)} \cong \bigoplus_{n \geq 0} \frac{I_n}{I_{n+1} + \sum_{j=2}^{i-1} x_j^j I_{n+1-j}}.$$

One can verify that

$$\begin{aligned} & ((I_{n+i} + \sum_{j=2}^{i-1} x_j^j I_{n+i-j}) : (x_i^i)^*) \\ &= (I_{n+i} : (x_i^i)^*) + \sum_{j=2}^{i-1} (x_j^j I_{n+i-j} : (x_i^i)^*) \quad [4, \text{Proposition 1.14}] \\ &= (I_{n+i} : (x_i^i)^*) + \sum_{j=2}^{i-1} x_j^j (I_{n+i-j} : (x_i^i)^*) \\ &= I_{n+1} + \sum_{j=2}^{i-1} x_j^j I_{n+1-j} \quad [\text{by Proposition 4.1}]. \end{aligned}$$

Hence  $(x_i^i)^*$  is  $\frac{G(\mathcal{F})}{((x_2^2)^*, \dots, (x_{i-1}^{i-1})^*)}$ -regular.  $\square$

**Proposition 4.7.** *Let  $2 \leq k \leq d$ . Then for all  $n \geq 1$ ,*

$$\ell \left( \frac{T'}{(I_n + (x_2^2, \dots, x_k^k))T'} \right) = \sum_{i=0}^{k-1} (-1)^i \left[ \sum_{1 \leq j_1 < \dots < j_i \leq k-1} \ell \left( \frac{T'}{(I_{n-(j_1+\dots+j_i)})T'} \right) \right].$$

*Proof.* The proof follows from Proposition 2.4 and Theorem 4.6.  $\square$

## 5. MONOMIAL GENERATORS OF $I_{n-1}$ MODULO $(I_n : x_d)$ AS A $\mathbb{k}$ -VECTOR SPACE

In this section we give an upper bound for the number of generators of  $I_{n-1}$  modulo  $(I_n : x_d)$  as a  $\mathbb{k}$ -vector space. We first show that  $(I_n : x_d) \subseteq I_{n-1}$ . Next we describe the generators of  $I_{n-1}$  modulo  $(I_n : x_d)$ .

The following lemma is simple, but we state it as it is crucially used to prove Lemma 5.2.

**Lemma 5.1.** (1) *Let  $1 \leq j \leq d-1$  and  $a \geq 1$ . Then*

$$(M_{j+1,d})^{(j+1)a} = x_{j+1}^{(j+1)a-j} (M_{j+1,d})^j + (M_{j+2,d})^{j+1} (M_{j+1,d})^{(j+1)(a-1)}.$$

(2) *Let  $1 \leq k < j \leq d-1$  and  $a, b \geq 1$ . Then*

$$(M_{k+1,d})^a (M_{j+1,d})^b = (M_{k+1,j+1})^a (M_{j+1,d})^b + (M_{k+1,d})^{a-1} (M_{j+2,d})^{b+1}.$$

*Proof.* (1) The proof follows by induction on  $a$ . (2) The proof follows by induction on  $a+b$ .  $\square$

Before we proceed we set up some notation. For  $1 \leq j \leq d-1$ , let  $a_j \neq 0$  and  $\mathbf{a}_j := (a_1, \dots, a_j) \in \mathbb{N}^j$ . We inductively define the set  $S(\mathbf{a}_j)$  as follows:

$$\begin{aligned} S(\mathbf{a}_1) &:= \{x_2^{2a_1-1}\} \\ S(\mathbf{a}_j) &:= \begin{cases} \{x_{j+1}^{(j+1)a_j-j}\} & \text{if } \{i < j | a_i \neq 0\} = \emptyset \\ \{x_{j+1}^{(j+1)a_j-j}\} S(\mathbf{a}_k) M_{k+1,j+1}^k & \text{if } \{i < j | a_i \neq 0\} \neq \emptyset \text{ and } k = \max\{i < j | a_i \neq 0\}. \end{cases} \end{aligned}$$

We set  $\mathbf{J}^{\mathbf{a}_j} := J_1^{a_1} \dots J_j^{a_j}$ . Let  $\text{wt } \mathbf{a}_j := a_1 + 2a_2 + \dots + ja_j$  be the weight of  $\mathbf{a}_j$ . For all  $n \in \mathbb{N}$  we define  $\Lambda_{j,n} := \{\mathbf{a}_j \in \mathbb{N}^j : \text{wt } \mathbf{a}_j = n, a_j \neq 0\}$ .

**Lemma 5.2.** *Let  $n \geq 2$ . Then*

- (1)  $(I_n : x_d) \subseteq I_{n-1}$ .
- (2) For all  $1 \leq j \leq d-1$  and for all  $\mathbf{a}_j \in \Lambda_{j,n-1}$ 
  - (a)  $S(\mathbf{a}_j) M_{j+1,d}^j \subseteq \mathbf{J}^{\mathbf{a}_j} \setminus \mathbf{m}' \mathbf{J}^{\mathbf{a}_j}$  where  $\mathbf{m}' = (x_2, \dots, x_d)T'$ .
  - (b) For all  $1 \leq j \leq d-1$ ,  $\mathbf{J}^{\mathbf{a}_j} \subseteq (S(\mathbf{a}_j) M_{j+1,d}^j) + (I_n : x_d)$ .
- (3)  $I_{n-1} = \sum_{j=1}^{d-1} \sum_{\mathbf{a}_j \in \Lambda_{j,n-1}} (S(\mathbf{a}_j) M_{j+1,d}^j) + (I_n : x_d)$ .

*Proof.* (1) By [4, Proposition 1.14] it is enough to show that for all  $j = 1 \dots d-1$  and  $\mathbf{a}_j \in \Lambda_{j,n}$ ,  $(\mathbf{J}^{\mathbf{a}_j} : x_d) \subseteq I_{n-1}$ . One can verify that

$$\begin{aligned}
 (\mathbf{J}^{\mathbf{a}_j} : x_d) &= (M_{2,d})^{2a_1} \dots (M_{j,d})^{ja_{j-1}} (M_{j+1,d})^{(j+1)a_j-1} \\
 &= (M_{2,d})^{2a_1} \dots [(M_{j,d})^{ja_{j-1}} (M_{j+1,d})^j] (M_{j+1,d})^{(j+1)a_j-(j+1)} \\
 &\subseteq (M_{2,d})^{2a_1} \dots (M_{j,d})^{j(a_{j-1}+1)} (M_{j+1,d})^{(j+1)(a_j-1)} \quad [\text{as } (M_{j+1,d}) \subseteq (M_{j,d})] \\
 &\subseteq I_{n-1},
 \end{aligned}$$

since  $a_1 + \dots + (j-2)a_{j-2} + (j-1)(a_{j-1}+1) + j(a_j-1) = n-1$ . This proves (1).

(2) Set  $r(\mathbf{a}_j) = \#\{i : 1 \leq i \leq j \text{ and } a_i \neq 0\}$ . We prove (2a) and (2b) by induction on  $r(\mathbf{a}_j)$ .

If  $r(\mathbf{a}_j) = 1$ , then  $S(\mathbf{a}_j) = \{x_{j+1}^{(j+1)a_j-j}\}$ . Hence  $S(\mathbf{a}_j)M_{j+1,d}^j = \{x_{j+1}^{(j+1)a_j-j}\}M_{j+1,d}^j \subseteq J_j^{a_j} = J_j^{\mathbf{a}_j}$ . As the degree of any monomial in the set  $S(\mathbf{a}_j)M_{j+1,d}^j$  is  $(j+1)a_j$ , it is not in the ideal  $\mathfrak{m}'J_j^{a_j}$ . Hence (2a) holds true for  $r(\mathbf{a}_j) = 1$ .

One can verify that

$$\begin{aligned}
 \mathbf{J}^{\mathbf{a}_j} &= (M_{j+1,d})^{(j+1)a_j} \\
 &= x_{j+1}^{(j+1)a_j-j} (M_{j+1,d})^j + (M_{j+2,d})^{j+1} (M_{j+1,d})^{(j+1)(a_j-1)} \quad [\text{by Lemma 5.1(1)}] \\
 &\subseteq S(\mathbf{a}_j)(M_{j+1,d})^j + (I_n : x_d)
 \end{aligned}$$

since

$$x_d(M_{j+2,d})^{j+1}(M_{j+1,d})^{(j+1)(a_j-1)} \subseteq J_{j+1}J_j^{a_j-1} \subseteq I_n,$$

and  $(j+1) + j(a_j-1) = ja_j + 1 = (n-1) + 1 = n$ . Hence (2b) is true for  $r(\mathbf{a}_j) = 1$ .

Now let  $r > 1$  and  $k = \max\{i | 1 \leq i < j \text{ and } a_i \neq 0\}$ . Then  $\mathbf{J}^{\mathbf{a}_j} = \mathbf{J}^{\mathbf{a}_k} J_j^{a_j}$  and

$$\begin{aligned}
 S(\mathbf{a}_j)M_{j+1,d}^j &= S(\mathbf{a}_k)M_{k+1,j+1}^k \left[ x_{j+1}^{(j+1)a_j-j} M_{j+1,d}^j \right] \\
 &\subseteq \mathbf{J}^{\mathbf{a}_k} J_j^{a_j} \quad [\text{by induction hypothesis}] \\
 &= \mathbf{J}^{\mathbf{a}_j}.
 \end{aligned}$$

Comparing the degree of the monomials in  $S(\mathbf{a}_j)M_{j+1,d}^j$  we conclude that these monomials are not in  $\mathfrak{m}'\mathbf{J}^{\mathbf{a}_j}$ . Hence (2a) holds true for  $r(\mathbf{a}_j) \geq 1$ .

Also

$$\begin{aligned}
& \mathbf{J}^{\mathbf{a}_j} \\
&= \mathbf{J}^{\mathbf{a}_k} J_j^{a_j} \\
&\subseteq \left( (S(\mathbf{a}_k) M_{k+1,d}^k) + (I_{n-j a_j} : x_d) \right) J_j^{a_j} \\
&\quad \text{[by induction hypothesis applied to } \mathbf{J}^{\mathbf{a}_k}] \\
&\subseteq x_{j+1}^{(j+1)a_j-j} (S(\mathbf{a}_k) M_{k+1,d}^k) (M_{j+1,d})^j + \left( S(\mathbf{a}_k) M_{k+1,d}^k \right) (I_{j a_j+1} : x_d) + (I_{n-j a_j} : x_d) J_j^{a_j} \\
&\quad \text{[by the case } r=1 \text{ applied to } J_j^{a_j}] \\
&\subseteq x_{j+1}^{(j+1)a_j-j} (S(\mathbf{a}_k) M_{k+1,d}^k) (M_{j+1,d})^j + (I_n : x_d) \quad \text{[by Lemma 5.2(2a)]} \\
&\subseteq x_{j+1}^{(j+1)a_j-j} (S(\mathbf{a}_k)) \left[ (M_{k+1,j+1})^k (M_{j+1,d})^j + (M_{k+1,d})^{k-1} (M_{j+2,d})^{j+1} \right] + (I_n : x_d) \quad \text{[by Lemma 5.1(2)]} \\
&= \left( S(\mathbf{a}_j) (M_{j+1,d})^j \right) + (I_n : x_d)
\end{aligned}$$

as

$$\begin{aligned}
& x_d (x_{j+1}^{(j+1)a_j-j} (S(\mathbf{a}_k)) (M_{k+1,d})^{k-1} (M_{j+2,d})^{j+1}) \\
&\subseteq \left[ (S(\mathbf{a}_k)) (x_{j+1} (M_{k+1,d})^{k-1}) \right] x_d (M_{j+2,d})^{j+1} (x_{j+1}^{(j+1)(a_j-1)}) \\
&\subseteq \mathbf{J}^{\mathbf{a}_k} J_{j+1} J_j^{a_j-1} \quad \text{[by Lemma 5.2(2a)]} \\
&\subseteq I_n.
\end{aligned}$$

This proves (2b) for all  $r(\mathbf{a}_j) \geq 1$ .

(3) The proof follows from (1) and (2).  $\square$

**Proposition 5.3.** *The set  $\{M + (I_n : x_d) \mid M \in \{\bigcup_{j=1}^{d-1} \bigcup_{\mathbf{a}_j \in \Lambda_{j,n-1}} \{S(\mathbf{a}_j) M_{j+1,d}^j\}\}\}$  generates  $\frac{I_{n-1}}{(I_n : x_d)}$  as a  $\mathbb{k}$ -vector space.*

*Proof.* Let  $M$  be a monomial in  $S(\mathbf{a}_j) M_{j+1,d}^j$ . By Lemma 5.2(2a),  $M \in \mathbf{J}^{\mathbf{a}_j}$ . Thus  $x_d x_i M \in (\mathfrak{m}')^2 J_1^{a_1} \cdots J_j^{a_j} = J_1^{a_1+1} \cdots J_j^{a_j} \subseteq I_n$ . This implies that  $x_i M \in (I_n : x_d)$  for all  $i = 2, \dots, d$ . Hence from Lemma 5.2(3), the monomials in  $S(\mathbf{a}_j) M_{j+1,d}^j$  generate  $I_{n-1}$  modulo  $(I_n : x_d)$  as a  $\mathbb{k}$ -vector space.  $\square$

From Proposition 5.3, giving an upper bound for the length of the vector space  $\frac{I_{n-1}}{(I_n : x_d)}$  involves counting monomials and hence it is combinatorial in nature. Hence we prove some preliminary results before we arrive at the main result of this section. We state the well known Vandermonde's identity which will be needed in our proofs.

**Lemma 5.4.** [Vandermonde's identity] Let  $n, r, s \in \mathbb{N}$ . Then

$$\sum_{i \geq 0} \binom{n}{i} \binom{s}{r-i} = \binom{n+s}{r}$$

The next lemma is the main step in proving our main result.

**Lemma 5.5.** *Fix  $1 \leq j \leq d-1$  and  $n > 1$ . Then*

$$\sum_{\mathbf{a}_j \in \Lambda_{j,n-1}} \#S(\mathbf{a}_j) = \binom{n-2}{j-1}.$$

*Proof.* We prove by induction on  $j$ . If  $j = 1$  then  $S(\mathbf{a}_1) = \{x_2^{2a_1-1}\}$ , and hence the assertion is true for  $j = 1$ .

Now let  $j > 1$ . Then

$$\begin{aligned} & \sum_{\mathbf{a}_j \in \Lambda_{j,n-1}} \#S(\mathbf{a}_j) \\ = & \begin{cases} \sum_{a_j=1}^{\lfloor \frac{n-2}{j} \rfloor} \sum_{i=1}^{j-1} \left[ \sum_{\mathbf{a}_i \in \Lambda_{i,n-1-ja_j}} \#S(\mathbf{a}_i) \right] \#M_{i+1,j+1}^i & \text{if } j \nmid (n-1) \\ \#S(0, \dots, 0, \frac{n-1}{j}) + \sum_{a_j=1}^{\lfloor \frac{n-2}{j} \rfloor} \sum_{i=1}^{j-1} \left[ \sum_{\mathbf{a}_i \in \Lambda_{i,n-1-ja_j}} \#S(\mathbf{a}_i) \right] \#M_{i+1,j+1}^i & \text{if } j \mid (n-1) \end{cases}. \end{aligned} \quad (5.6)$$

Define

$$\alpha_{j,n} := \begin{cases} 0 & \text{if } j \nmid (n-1) \\ 1 & \text{if } j \mid (n-1) \end{cases}.$$

Then (5.6) can be written as

$$\begin{aligned} & \sum_{\mathbf{a}_j \in \Lambda_{j,n-1}} \#S(\mathbf{a}_j) \\ = & \alpha_{j,n} + \sum_{a_j=1}^{\lfloor \frac{n-2}{j} \rfloor} \sum_{i=1}^{j-1} \left[ \sum_{\mathbf{a}_i \in \Lambda_{i,n-1-ja_j}} \#S(\mathbf{a}_i) \right] \#M_{i+1,j+1}^i \\ = & \alpha_{j,n} + \sum_{a_j=1}^{\lfloor \frac{n-2}{j} \rfloor} \sum_{i=1}^{j-1} \binom{n-ja_j-2}{i-1} \binom{j}{j-i} & \text{[by induction hypothesis]} \\ = & \alpha_{j,n} + \sum_{a_j=1}^{\lfloor \frac{n-2}{j} \rfloor} \left[ \sum_{i=0}^{j-1} \binom{n-ja_j-2}{i} \binom{j}{j-i-1} \right] - \binom{n-ja_j-2}{j-1} & \text{[replacing } i-1 \text{ by } i] \\ = & \alpha_{j,n} + \sum_{a_j=1}^{\lfloor \frac{n-2}{j} \rfloor} \left[ \binom{n-j(a_j-1)-2}{j-1} - \binom{n-ja_j-2}{j-1} \right] & \text{[by Lemma 5.4]} \\ = & \alpha_{j,n} + \binom{n-2}{j-1} - \alpha_{j,n} \\ = & \binom{n-2}{j-1}. \end{aligned}$$



□

We are now ready to prove the main result in this section.

**Proposition 5.7.** *Let  $d, n \geq 2$ . Then*

$$\ell\left(\frac{I_{n-1}}{(I_n : x_d)}\right) \leq \binom{n+d-3}{d-2}.$$

*Proof.* By Proposition 5.3 we get

$$\begin{aligned} \ell\left(\frac{I_{n-1}}{(I_n : x_d)}\right) &\leq \sum_{j=1}^{d-1} \left[ \sum_{\mathbf{a}_j \in \Lambda_{j,n-1}} \#S(\mathbf{a}_j) \right] \#M_{j+1,d}^j \\ &= \sum_{j=1}^{d-1} \binom{n-2}{j-1} \binom{d-1}{d-j-1} && \text{[by Lemma 5.5].} \\ &= \sum_{i=0}^{d-2} \binom{n-2}{i} \binom{d-1}{d-i-2} && \text{[put } i = j-1] \\ &= \binom{n+d-3}{d-2} && \text{[by Lemma 5.4].} \end{aligned}$$

□

## 6. COHEN-MACAULAYNESS OF $R/(\mathfrak{p}^{(n)} + (\mathbf{f}_k))$

In [5, Proposition 7.6] Goto showed that  $R/(\mathfrak{p}^{(n)} + (\mathbf{f}_{d-1}))$  is Cohen-Macaulay for  $d = 3, 4$  and  $n \leq \binom{d-1}{2}$ . This was done by explicitly describing  $\mathfrak{p}^{(n)}$  for  $d \leq 4$  and  $n \leq \binom{d-1}{2}$ . In this section, we generalise Goto's result for all  $d \geq 2$  and  $n \geq 1$ . A lower bound for  $\ell(R/(\mathfrak{p}^{(n)} + (\mathbf{f}_k, x_1)))$  was given using the multiplicity formula (Theorem 2.7). In this section, we show that the inequality in Theorem 2.7 is indeed an equality (Theorem 6.5). This implies that for all  $n \geq 1$  and  $1 \leq k \leq d-1$ , the rings  $R/(\mathfrak{p}^{(n)} + (\mathbf{f}_k))$  are Cohen-Macaulay.

As a consequence, we describe  $\mathfrak{p}^{(n)}$  for all  $d \geq 2$  and all  $n \geq 1$ . In particular we prove that  $\mathfrak{p}^{(n)} = \mathcal{I}_n R$  and  $LI(\mathfrak{p}^{(n)} T') = I_n T'$  for all  $d \geq 2$  and  $n \geq 1$ .

We first give an upper bound on  $\ell(T'/I_n)$ . This is crucial to prove an interesting result which shows that the equality of the lengths of the various modules (over different rings) in Theorem 6.2.

**Proposition 6.1.** *Let  $d \geq 2$ . Then for all  $n \geq 1$ ,*

$$\ell\left(\frac{T'}{I_n}\right) \leq d \binom{n+d-2}{d-1}.$$

*Proof.* We prove by double induction on  $n$  and  $d$ . If  $n = 1$ , then

$$\ell\left(\frac{T'}{I_1}\right) = \ell\left(\frac{k[x_2, \dots, x_d]}{(x_2, \dots, x_d)^2}\right) = d.$$

If  $d = 2$ , then

$$\ell\left(\frac{T'}{I_n}\right) = \ell\left(\frac{k[x_2]}{(x_2)^{2n}}\right) = 2n.$$

Now let  $n > 1$  and  $d > 2$ . From the exact sequence

$$0 \rightarrow \frac{T'}{(I_n : x_d)} \xrightarrow{\cdot x_d} \frac{T'}{I_n} \rightarrow \frac{T'}{I_n + (x_d)} \rightarrow 0$$

we get

$$\begin{aligned} & \ell\left(\frac{T'}{I_n}\right) \\ = & \ell\left(\frac{T'}{I_n + (x_d)}\right) + \ell\left(\frac{T'}{(I_n : x_d)}\right) \\ = & \ell\left(\frac{T'}{I_n + (x_d)}\right) + \ell\left(\frac{T'}{I_{n-1}}\right) + \ell\left(\frac{I_{n-1}}{(I_n : x_d)}\right) \quad [\text{Lemma 5.2(1)}] \\ \leq & (d-1)\binom{n+d-3}{d-2} + d\binom{n-1+d-2}{d-1} + \binom{n+d-3}{d-2} \quad [\text{by induction hypothesis and Proposition 5.7}] \\ = & d\binom{n+d-3}{d-2} + d\binom{n+d-3}{d-1} \\ = & d\binom{n+d-2}{d-1}. \end{aligned}$$

□

**Theorem 6.2.** *Let  $d \geq 2$ . Then for all  $n \geq 1$ ,*

$$\begin{aligned} e\left(x_1; \frac{R}{\mathfrak{p}^{(n)}}\right) &= \ell\left(\frac{R}{\mathfrak{p}^{(n)} + (x_1)}\right) = \ell_R\left(\frac{R}{(\mathcal{I}_n, x_1)R}\right) = \ell_{T'}\left(\frac{T'}{\mathcal{I}_n T'}\right) \\ &= \ell_{T'}\left(\frac{T'}{LI(\mathcal{I}_n)T'}\right) = \ell\left(\frac{T'}{I_n}\right) = d\binom{n+d-2}{d-1}. \end{aligned}$$

*Proof.* From Proposition 2.13(1)  $\mathcal{I}_n R \subseteq \mathfrak{p}^{(n)}$ . Hence,

$$\begin{aligned} e\left(x_1; \frac{R}{\mathfrak{p}^{(n)}}\right) &= \ell_R\left(\frac{R}{\mathfrak{p}^{(n)} + (x_1)}\right) \quad [\text{as } R/\mathfrak{p}^{(n)} \text{ is Cohen-Macaulay}] \\ &\leq \ell_R\left(\frac{R}{(\mathcal{I}_n, x_1)R}\right). \end{aligned} \tag{6.3}$$

By Proposition 2.13(3), for any prime  $\mathfrak{q} \neq \mathfrak{m}$ ,  $((\mathcal{I}_n, x_1)T)_{\mathfrak{q}} = T$ . This implies that  $\text{Supp}_T\left(\frac{T}{(\mathcal{I}_n, x_1)T}\right) = \{\mathfrak{m}\}$ . Hence we get

$$\begin{aligned}
\ell_R \left( \frac{R}{(\mathcal{I}_n, x_1)R} \right) &= \ell_{T'} \left( \frac{T'}{\mathcal{I}_n T'} \right) && [\text{Lemma 2.8}] \\
&= \ell_{T'} \left( \frac{T'}{LI(\mathcal{I}_n)T'} \right) && [1, \text{Proposition 2.1}] \\
&\leq \ell_{T'} \left( \frac{T'}{I_n} \right) && [\text{Proposition 3.4}] \\
&\leq d \binom{n+d-2}{d-1} && [\text{Proposition 6.1}] \\
&= e \left( x_1; \frac{R}{\mathfrak{p}} \right) \ell_{R_{\mathfrak{p}}} \left( \frac{R_{\mathfrak{p}}}{\mathfrak{p}^n R_{\mathfrak{p}}} \right) \\
&= e \left( x_1; \frac{R}{\mathfrak{p}} \right) \ell_{R_{\mathfrak{p}}} \left( \frac{R_{\mathfrak{p}}}{\mathfrak{p}^{(n)} R_{\mathfrak{p}}} \right) && [\text{since } \mathfrak{p}^{(n)} R_{\mathfrak{p}} = \mathfrak{p}^n R_{\mathfrak{p}}] \\
&= e \left( x_1; \frac{R}{\mathfrak{p}^{(n)}} \right) && [\text{by [10, Theorem 14.7]}]. \tag{6.4}
\end{aligned}$$

Thus equality holds in (6.3) and (6.4) which proves the theorem.  $\square$

**Theorem 6.5.** *Let  $d \geq 2$  and  $1 \leq k \leq d-1$ . Let  $\mathbf{f}_k$  be as in (2.2). Then for all  $n \geq 1$ ,*

$$\begin{aligned}
e \left( x_1; \frac{R}{\mathfrak{p}^{(n)} + (\mathbf{f}_k)} \right) &= \ell_R \left( \frac{R}{\mathfrak{p}^{(n)} + (x_1, \mathbf{f}_k)} \right) = \ell_{T'} \left( \frac{T'}{(\mathcal{I}_n + \mathbf{f}_k)T'} \right) \\
&= \ell_{T'} \left( \frac{T'}{LI((\mathcal{I}_n + \mathbf{f}_k)T')} \right) = \ell \left( \frac{T'}{I_n + (x_2^2, \dots, x_{k+1}^{k+1})} \right) \\
&= d \sum_{i=0}^k (-1)^i \left[ \sum_{1 \leq j_1 < \dots < j_i \leq k} \binom{n - (j_1 + \dots + j_i) + d - 2}{d-1} \right].
\end{aligned}$$

In particular,  $\frac{R}{\mathfrak{p}^{(n)} + (\mathbf{f}_k)}$  is Cohen-Macaulay.

*Proof.* From Proposition 2.13(1)  $(\mathcal{I}_n, x_1, \mathbf{f}_k)R \subseteq (\mathfrak{p}^{(n)}, x_1, \mathbf{f}_k)R$ . Hence

$$\begin{aligned}
e \left( x_1; \frac{R}{\mathfrak{p}^{(n)} + (\mathbf{f}_k)} \right) &\leq \ell_R \left( \frac{R}{\mathfrak{p}^{(n)} + (x_1, \mathbf{f}_k)} \right) && [10, \text{Theorem 14.10}] \\
&\leq \ell_R \left( \frac{R}{(\mathcal{I}_n, x_1, \mathbf{f}_k)R} \right). \tag{6.6}
\end{aligned}$$

Since  $(\mathcal{I}_n, x_1)T \subseteq (\mathcal{I}_n, \mathbf{f}_k, x_1)T$  by Proposition 2.13(3), for any prime  $\mathfrak{q} \neq \mathfrak{m}$ ,  $((\mathcal{I}_n, \mathbf{f}_k, x_1)T)_{\mathfrak{q}} = T$ . This implies that  $\text{Supp}_T \left( \frac{T}{(\mathcal{I}_n, \mathbf{f}_k, x_1)T} \right) = \{\mathfrak{m}\}$ . Hence we get

$$\begin{aligned}
& \ell_R \left( \frac{R}{(\mathcal{I}_n, x_1, \mathbf{f}_k)R} \right) \\
&= \ell_R \left( \frac{T}{(\mathcal{I}_n, x_1, \mathbf{f}_k)T} \otimes_T R \right) \\
&= \ell_T \left( \frac{T}{(\mathcal{I}_n, x_1, \mathbf{f}_k)T} \right) \quad [\text{Lemma 2.8}] \\
&= \ell_{T'} \left( \frac{T'}{(\mathcal{I}_n, \mathbf{f}_k)T'} \right) \\
&= \ell_{T'} \left( \frac{T'}{LI((\mathcal{I}_n, \mathbf{f}_k)T')} \right) \quad [1, \text{Proposition 2.1}] \\
&\leq \ell_{T'} \left( \frac{T'}{I_n + (x_2^2, \dots, x_{k+1}^{k+1})} \right) \quad [\text{Propositions 3.4 and 3.6(1)}] \\
&= \sum_{i=0}^k (-1)^i \left[ \sum_{1 \leq j_1 < \dots < j_i \leq k} \ell \left( \frac{T'}{(I_{n-(j_1+\dots+j_i)})T'} \right) \right] \quad [\text{Proposition 4.7}] \\
&= d \sum_{i=0}^k (-1)^i \left[ \sum_{1 \leq j_1 < \dots < j_i \leq k} \binom{n - (j_1 + \dots + j_i) + d - 2}{d - 1} \right] \quad [\text{Theorem 6.2}] \\
&= d \sum_{i=0}^k (-1)^i \left[ \sum_{1 \leq j_1 < \dots < j_i \leq k} \ell \left( \frac{R_{\mathfrak{p}}}{\mathfrak{p}^{n-[j_1+\dots+j_i]} R_{\mathfrak{p}}} \right) \right] \\
&= e \left( x_1; \frac{R}{\mathfrak{p}^{(n)} + (\mathbf{f}_k)} \right) \quad [[5, \text{Proposition 5.3(3)}] \text{ and Corollary 2.6(1)}]. \quad (6.7)
\end{aligned}$$

Hence equality holds in (6.6) and (6.7) which proves the theorem.  $\square$

We end this section by explicitly describing the generators of  $\mathfrak{p}^{(n)}$  for all  $n \geq 1$ . We also describe the leading ideal  $LI(\mathfrak{p}^{(n)}T')$ .

**Theorem 6.8.** (1) For all  $n \geq 1$ ,  $\mathfrak{p}^{(n)} = \mathcal{I}_n R$ .

(2) For all  $n \geq d$ ,  $\mathfrak{p}^{(n)} = \sum_{a_1+2a_2+\dots+(d-1)a_{d-1}=n} \mathfrak{p}^{a_1} (\mathfrak{p}^{(2)})^{a_2} \dots (\mathfrak{p}^{(d-1)})^{a_{d-1}}$ .

(3) For all  $n \geq 1$ ,  $LI(\mathfrak{p}^{(n)}T') = I_n = LI(\mathcal{I}_n T')$ .

*Proof.* (1) By Theorem 6.2 we get

$$\ell \left( \frac{R}{\mathfrak{p}^{(n)} + (x_1)} \right) = \ell \left( \frac{R}{\mathcal{I}_n R + (x_1)} \right).$$

This implies that  $\mathfrak{p}^{(n)} = \mathcal{I}_n R + x_1(\mathfrak{p}^{(n)} : (x_1))$ . As  $x_1$  is a nonzerodivisor on  $R/\mathfrak{p}^{(n)}$ ,  $(\mathfrak{p}^{(n)} : (x_1)) = \mathfrak{p}^{(n)}$ . By Nakayama's lemma,  $\mathfrak{p}^{(n)} = \mathcal{I}_n R$ .

(2) For all  $n \geq d$ ,

$$\begin{aligned}
\mathfrak{p}^{(n)} &= \mathcal{I}_n R \\
&= \sum_{a_1+2a_2+\dots+(d-1)a_{d-1}=n} \mathcal{J}_1^{a_1} \mathcal{J}_2^{a_2} \dots \mathcal{J}_{d-1}^{a_{d-1}} R \\
&\subseteq \sum_{a_1+2a_2+\dots+(d-1)a_{d-1}=n} \mathfrak{p}^{a_1} (\mathfrak{p}^{(2)})^{a_2} \dots (\mathfrak{p}^{(d-1)})^{a_{d-1}} \quad [\text{by Proposition 2.13(1)}] \\
&\subseteq \mathfrak{p}^{(n)}.
\end{aligned}$$

Hence equality holds.

(3) The proof follows from Proposition 3.4 and Theorem 6.2.  $\square$

## 7. COHEN-MACAULAYNESS AND GORENSTEINNESS OF SYMBOLIC BLOWUP ALGEBRAS

In this section we show that both symbolic blowup algebras  $G_s(\mathfrak{p})$  and  $\mathcal{R}_s(\mathfrak{p})$  are Cohen-Macaulay. We also show that  $G_s(\mathfrak{p})$  is Gorenstein and find necessary and sufficient conditions for  $\mathcal{R}_s(\mathfrak{p})$  to be Gorenstein.

From [5, Theroem 6.7] and Theorem 6.5, it follows that  $G_s(\mathfrak{p})$  is Cohen-Macaulay. In the following theorem we give an alternate argument for  $G_s(\mathfrak{p})$  to be Cohen-Macaulay. Put  $f_0 = x_1$ . Let  $f_i$ 's be as in (2.2) and let  $f_i^*$  denotes the image of  $f_i$  in  $\mathfrak{p}^{(i)}/\mathfrak{p}^{(i+1)}$ . In [5, Proposition 5.3], Goto showed that  $\mathbf{f}_{d-1}$  is a homogenous system of parameters in  $G_s(\mathfrak{p})$ . We show that  $f_0^*, \mathbf{f}_{d-1}^*$  is a regular sequence in  $G_s(\mathfrak{p})$ .

**Theorem 7.1.** *Let  $d \geq 2$ . Then*

- (1)  $G_s(\mathfrak{p})$  is Cohen-Macaulay.
- (2)  $G_s(\mathfrak{p})$  is Gorenstein.

*Proof.* (1) By induction on  $k$ , we prove that  $f_0^*, \mathbf{f}_k^*$  is a regular sequence in  $G_s(\mathfrak{p})$  for all  $k = 0, \dots, d-1$ . Let  $k = 0$ . Then as  $x_1$  is a nonzerodivisor on  $R/\mathfrak{p}^{(n)}$  for all  $n$ , we conclude that  $f_0^*$  is a nonzerodivisor in  $G_s(\mathfrak{p})$ . Now let  $k \geq 1$  and assume that  $f_0^*, \mathbf{f}_{k-1}^*$  is a regular sequence in  $G_s(\mathfrak{p})$ . Then

$$\frac{G_s(\mathfrak{p})}{(f_0^*, \mathbf{f}_{k-1}^*)} \cong \bigoplus_{n \geq 0} \frac{\mathfrak{p}^{(n)}}{\mathfrak{p}^{(n+1)} + \sum_{j=0}^{k-1} f_j \mathfrak{p}^{(n-j)}} \cong \bigoplus_{n \geq 0} \frac{\mathfrak{p}^{(n)} + (f_0, \mathbf{f}_{k-1})}{\mathfrak{p}^{(n+1)} + (f_0, \mathbf{f}_{k-1})}$$

Hence, to show that  $f_k^*$  is a nonzerodivisor on  $\frac{G_s(\mathfrak{p})}{(f_0^*, \mathbf{f}_{k-1}^*)}$  it is enough to show that  $((\mathfrak{p}^{(n+1)}, f_0, \mathbf{f}_{k-1}) : (f_k)) = (\mathfrak{p}^{(n+1-k)}, f_0, \mathbf{f}_{k-1})$ . Since

$$\begin{aligned}
\ell\left(\frac{R}{((\mathfrak{p}^{(n+1)}, f_0, \mathbf{f}_{k-1}) : (f_k))}\right) &= \ell\left(\frac{R}{(\mathfrak{p}^{(n+1)}, f_0, \mathbf{f}_{k-1})}\right) - \ell\left(\frac{R}{(\mathfrak{p}^{(n+1)}, f_0, \mathbf{f}_k)}\right) \\
&= \ell\left(\frac{T'}{I_{n+1} + (x_2^2, \dots, x_k^k)}\right) - \ell\left(\frac{T'}{I_{n+1} + (x_2^2, \dots, x_{k+1}^{k+1})}\right) \quad [\text{Theorem 6.5}] \\
&= \ell\left(\frac{T'}{(I_{n+1} + (x_2^2, \dots, x_k^k)) : x_{k+1}^{k+1}}\right) \\
&= \ell\left(\frac{T'}{I_{n+1-k} + (x_2^2, \dots, x_k^k)}\right) \quad [\text{Proposition 4.1}] \\
&= \ell\left(\frac{R}{(\mathfrak{p}^{(n+1-k)}, f_0, \mathbf{f}_{k-1})}\right) \quad [\text{Theorem 6.5}]
\end{aligned}$$

we get  $((\mathfrak{p}^{(n+1)}, f_0, \mathbf{f}_{k-1}) : (f_k)) = (\mathfrak{p}^{(n+1-k)}, f_0, \mathbf{f}_{k-1})$ . Hence  $f_k$  is a nonzerodivisor in  $G_s(\mathfrak{p})/(f_0^*, \mathbf{f}_{k-1}^*)$ . This proves (1).

(2) As  $G(\mathfrak{p}R_{\mathfrak{p}})$  is a polynomial ring, it is Gorenstein. Hence the result follows from Theorem 6.5 and [5, Corollary 5.8].  $\square$

**Theorem 7.2.** *Let  $d \geq 2$ . Then*

- (1)  $\mathcal{R}_s(\mathfrak{p}) = R[\mathfrak{p}t, \mathcal{J}_2t^2, \dots, \mathcal{J}_{d-1}t^{d-1}]$ .
- (2)  $\mathcal{R}_s(\mathfrak{p})$  is Cohen-Macaulay.
- (3)  $\mathcal{R}_s(\mathfrak{p})$  is Gorenstein if and only if  $d = 3$ .

*Proof.* (1) The proof follows from Theorem 6.8(2).

(2) By [5, Theorem 6.7], it suffices to show that  $\frac{R}{\mathfrak{p}^{(n)} + (\mathbf{f}_{d-1})}$  is Cohen-Macaulay for  $1 \leq n \leq \binom{d-1}{2}$ . This holds true by Theorem 6.5.

(3) By [5, Lemma 6.1], the a-invariant of  $(G_s(\mathfrak{p}))$ ,  $a(G_s(\mathfrak{p})) = -(d-1)$ . By [5, Theorem 6.6], and Theorem 7.1,  $\mathcal{R}_s(\mathfrak{p})$  is Gorenstein if and only if  $d = 3$ .  $\square$

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